

On exact singular wave functions of two identical particles

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Abstract. We discuss the occurrence and properties of exact singular anyonic wave functions describing stationary states of two identical charged particles moving on a plane and under the influence of a perpendicular uniform magnetic field.

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1 Introduction

The general principles of quantum mechanics requires that localized stationary states of a system be represented by square integrable functions so as to be consistent with the probabilistic interpretation of Born. This is in particular the situation for the motion in a potential well rising indefinitely at infinity and at a singular point localized, say at the origin. Here we are concerned with this condition in two space dimensions. Let us suppose that the wave function Ψ behaves as $r^s u(r)$ near the origin where $u(r)$ is a well-behaved function of r , the distance from the origin, for $r \rightarrow 0$. Then for $\int |\Psi|^2 d^2r$, to be convergent in a neighborhood of the origin, one should have $s > -1$. There is however a second condition stating that the origin as a singular point should not be a source nor sink for matter [1]. The outward flow of the probability flux \mathbf{J} through a small sphere of radius r must vanish as $r \rightarrow 0$:

$$\oint \mathbf{J} \cdot d\mathbf{a} = -\frac{i\hbar}{2m} r^{2s+1} \int_0^{2\pi} \left(u^* \frac{\partial u}{\partial r} - u \frac{\partial u^*}{\partial r} \right) d\theta \rightarrow 0$$

m being the particle mass. This puts a stronger bound on s : *i.e.* $s > -\frac{1}{2}$. However if u is a real function this condition is always satisfied as it is in the case we consider now.

In most soluble two-dimensional problems presenting cylindrical symmetric quantization of angular momentum to values $\hbar l$ leads to the condition $s^2 = l^2$ or $s = \pm |l|$ equivalently. For the wave function to be single-valued one imposes $l \in \mathbb{Z}$ and the choice $s = -|l|$ is excluded since it is inconsistent with $s > -1$. But with the discovery of anyonic behavior for wave functions of identical planar particles by Leinaas and Myrheim [2], l as quantized relative angular momentum of two particles is a priori a real number: $l \in \mathbb{R}$. Then it is possible to consider the second option $s = -|l|$, so long as $s > -1$. This possibility

which has been pointed out a decade ago by Grundberg *et al.* [3] in a general context leads to singular anyonic wave functions.

In this short note, we present examples of such functions occurring in the relative motion of two identical planar particles of mass m and charge e in a perpendicular uniform magnetic field B . First we show how to construct the wave functions in terms of Biconfluent Heun Polynomials and how quantization conditions are imposed. In the next section we discuss the structure of the low-lying singular wave functions. Finally we close the paper with some comments and physical interpretations on this unusual behavior.

2 Construction of the wave functions

As the Coulomb repulsion between the particles has rotational symmetry, we use cylindrical coordinates and the symmetric gauge to describe the magnetic field. The Hamiltonian of this relative motion reads:

$$H = \frac{1}{m} \left[p_r^2 + \left(\frac{p_\theta}{r} + \frac{eB}{2} r \right)^2 \right] + \kappa \frac{e^2}{r}. \quad (1)$$

Here, r is the particle separation, θ its inclination with respect to a reference axis, p_r and p_θ are the conjugate momenta and $\kappa = \frac{1}{4\pi\epsilon_0}$ in SI units.

The wave functions of stationary states $\Psi(r, \theta)$ are solutions of the Schrödinger equation $H\Psi = E\Psi$ with energy E . As p_θ is conserved and in view of possible exotic statistics in two dimensions we may set $\Psi(r, \theta) \simeq e^{il\theta} R_l(r)$ with $l \in \mathbb{R}$. To put forward the competition between magnetic length l_B and the Bohr radius l_0 defined by:

$$l_B^2 = \frac{2\hbar}{eB} \quad \text{and} \quad l_0 = \frac{\hbar^2}{\kappa e^2 m} \quad (2)$$

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we recast the radial Schrödinger equation for $R_l(r)$ in its dimensionless form:

$$R_l'' + \frac{1}{\xi} R_l' - \left[\frac{l^2}{\xi^2} + \frac{C}{\xi} - \xi^2 - 2(\epsilon - l) \right] R_l = 0 \quad (3)$$

using the scaled variables ξ , ϵ and C given by:

$$E = \frac{\hbar^2}{m l_B^2} \epsilon \quad r = \xi l_B \sqrt{2} \quad C = \frac{l_B}{l_0} \sqrt{2}. \quad (4)$$

From equation (3), it can be seen that the asymptotic form of $R_l(\xi)$ is $e^{-\frac{\xi^2}{2}}$. We set $R_l(\xi) = v_l(\xi) e^{-\frac{\xi^2}{2}}$ with the assumption that $v_l(\xi)$ is of milder growth, at $\xi \rightarrow \infty$, than $e^{-\frac{\xi^2}{2}}$. $v_l(\xi)$ fulfills now an ‘‘associated’’ Biconfluent Heun equation:

$$\xi v_l'' + (1 - 2\xi^2) v_l' - \{C + 2(\epsilon - l - 1)\xi\} v_l - \frac{l^2}{\xi} v_l = 0 \quad (5)$$

with Maroni’s canonical parameters $\alpha = \beta = 0$, $\gamma = 2(\epsilon - l)$ and $\delta = 2C$ [4]. We seek now a power series solution of the form $v_l(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi^{s+n}$. Substitution of this series into equation (5) yields the recursion relations:

$$\begin{aligned} (s^2 - l^2) \alpha_0 &= 0 \\ [(s+1)^2 - l^2] \alpha_1 &= C \alpha_0, \\ &\vdots \\ [(s+n)^2 - l^2] \alpha_n &= \\ C \alpha_{n-1} + 2[n+s-\epsilon+l-1] \alpha_{n-2}. \end{aligned} \quad (6)$$

Assuming $\alpha_0 \neq 0$, we have $s^2 = l^2$ or $s = \sigma|l|$, with $\sigma = \pm$. Now posing $\alpha_n = A_n^{(\sigma)} \frac{1}{(1+2s)_n n!}$ we recover the canonical recursion of the Biconfluent Heun Function $N(2s, 0, 2(\epsilon - l), 2C; \xi)$ [4]:

$$\begin{aligned} A_{n+2}^{(\sigma)} &= C A_{n+1}^{(\sigma)} \\ -2(n+1)(n+1+2s)(\epsilon - n - 1 - l - s) A_n^{(\sigma)}, \end{aligned} \quad (7)$$

with $A_0^{(\sigma)} = 1$ and $A_1^{(\sigma)} = C$; the parameter s may be $+|l|$ or $-|l|$ and label regular or singular wave functions near the origin.

As $N(2s, 0, 2(\epsilon - l), 2C; \xi)$ grows as e^{ξ^2} [4] it is necessary to cut its defining power series down to a polynomial by imposing two conditions on the recursion relation (7):

- energy quantization $\epsilon = \epsilon_n = (n + 1 + l + s)$,
- and $A_{n+1}^{(\sigma)}(C, l) = 0$.

In general for given B (or C) the angular momentum $l = l_k$ is the k th solution of an algebraic equation of order n for n odd and of order $(n - 1)$ for n even. This has been already pointed out in [5]. The main consequence is that the magnetic field fixes l and thus gives rise to the expected anyonic behavior. Note that the effective radial potential becomes also modified for each eigenstate since it depends on l .

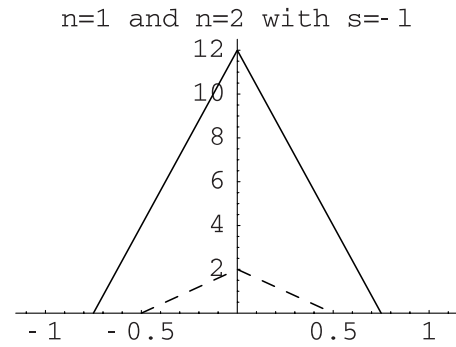


Fig. 1. C^2 versus l for $n = 1$ (broken line) and $n = 2$ (continuous line).

Globally the wave function of a stationary state reads:

$$\psi_{n,l_k}^{(\sigma)}(\theta, \xi) \simeq e^{il_k \theta} \xi^{\sigma|l_k|} \Pi_n(\xi, C) e^{-\frac{\xi^2}{2}} \quad (8)$$

where $\Pi_n(\xi, C)$ is a special Biconfluent Heun polynomial, obtained by truncating the power series.

As well-behaved wave functions with $\sigma = +$ have been already discussed in [5], we shall be concerned here with the low-lying $\psi_{n,l_k}^{(-)}$ with $|l_k| < 1$. The corresponding energy eigenvalues are $\epsilon_{n,l_k}^{(-)} = n + 1 + l_k - |l_k|$. In the following we shall assume that $C \neq 0$ (no infinitely strong magnetic field).

3 Properties of the wave functions

They appear to be similar for pairs of the quantum number n . This is essentially due to the fact that the conditions $A_n^{(\sigma)} = 0$ as well as $A_{n+1}^{(\sigma)} = 0$ for n even lead to a pair of polynomials of same order in C^2 since $A_{n+1}^{(\sigma)}$ contains an overall factor C . For $\sigma = -$, we shall study the variation of C^2 as a function of l in the interval $-1 < l < 1$. Two possible behaviors of the radial probability density for the relative separation r arise: divergent for $\frac{1}{2} < |l| < 1$ and convergent for $0 < |l| < \frac{1}{2}$ in the neighborhood of zero. This is the only physical relevant quantity since rotational symmetry yields a uniform probability density for the inclination angle θ .

The first group of states is related to $n = 1, 2$. The vanishing of $A_{n+1}^{(\sigma)} = 0$ yields linear relations between C^2 and l . For $n = 1, 2$, the equations are respectively $C^2 = 2(1 - 2|l|)$ and $C^2 = 4(3 - 4|l|)$. This is shown in Figure 1. We have given the probability distribution of states $r|\psi_{n,l_k}^{(-)}(\theta, r)|^2$ in Figures 2 and 3 as function of r .

$$\begin{aligned} P_1^{(-)}(r, C) &\simeq \left(\frac{r}{l_B \sqrt{2}} \right)^{\frac{C^2}{2}} \left(1 + \frac{2}{C} \frac{r}{l_B \sqrt{2}} \right)^2 e^{-\frac{r^2}{2l_B^2}} \\ P_2^{(-)}(r, C) &\simeq \left(\frac{r}{l_B \sqrt{2}} \right)^{\frac{C^2}{8} - \frac{1}{2}} \\ &\times \left(1 + \frac{8r}{\sqrt{2} l_B (C^2 - 4)} + \frac{2r^2}{2l_B^2 (C^2 - 4)} \right)^2 e^{-\frac{r^2}{2l_B^2}}. \end{aligned} \quad (9)$$

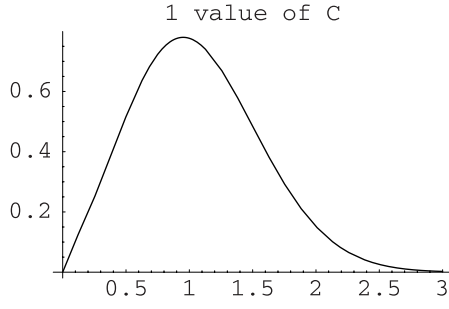


Fig. 2. Probability distribution $P_1^{(-)}(r, C = 1)$ for $n = 1$.

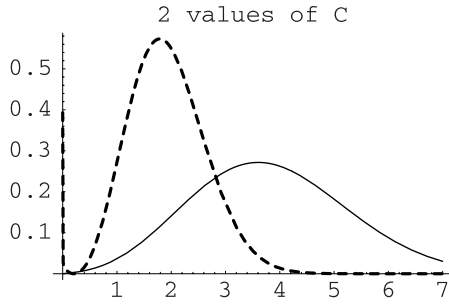


Fig. 3. Probability distributions $P_2^{(-)}(r, C)$ for $n = 2$, with $C = 2\sqrt{2}$ (continuous line) and $C = \sqrt{2}$ (broken line).

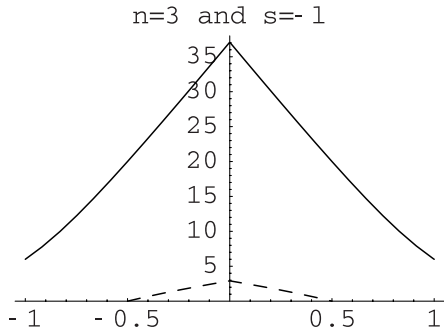


Fig. 4. C^2 versus of l respectively for $n = 3$, the upper (resp. lower) branch correspond to + (resp. -) in expression of C^2 .

To save space, normalization factors with involved expressions are not given but are implicitly taken into account. In particular, we see that at some value of $C < 2$ for $n = 2$ the radial probability distribution starts to develop an integrable singularity at the origin.

The next group of states is given by $n = 3, 4$. The conditions fixing l in terms of C are $A_4^{(-)}(C, l) = 0$ and $A_5^{(-)}(C, l) = 0$, which correspond to the equations $C^2 = 20(1 - |l|) \pm \sqrt{64|l|^2 - 28|l| + 73}$ and $C^2 = (50 - 40|l|) \pm 6\sqrt{16|l|^2 - 40|l| + 33}$. Since $C = 0$ is discarded we obtain quadratic equations which yield two values $l_k, k = 1, 2$. In Figure 4, (resp. 6) we plot C^2 in terms of l for $n = 3$, (resp. 4):

We observe that for certain values of the magnetic field such that C^2 is in a gap:

$$- (20 - 2\sqrt{73}) < C^2 < 6 \quad \text{for } n = 3$$

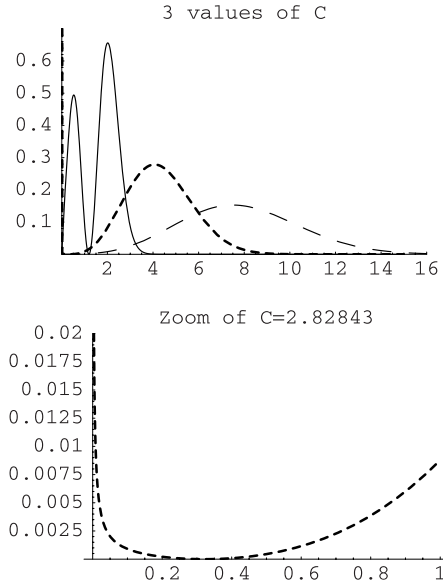


Fig. 5. Probability distributions $P_3^{(-)}(r, C)$ for $n = 3$.

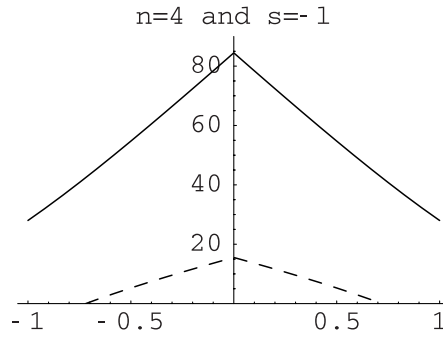


Fig. 6. C^2 versus of l respectively for $n = 4$, the upper (resp. lower) branch correspond to + (resp. -) in expression of C^2 .

$$- (50 - 6\sqrt{33}) < C^2 < 28 \quad \text{for } n = 4$$

there can be no states $\Psi_{n,l_k}^{(-)}(\theta, r)$ for $n = 3, 4$. The behavior of the probability distributions depends on whether $|l|$ is larger or smaller than $\frac{1}{2}$.

For $n = 3$ we may have 3 types of curves (see Fig. 5):

- $P_3^{(-)}(r, C = 5)$ (broken line) and $P_3^{(-)}(r, C = 1)$ (continuous line) have smooth behavior at the origin;
- $P_3^{(-)}(r, C = 2\sqrt{2})$ (dotted curve, with magnified part in lower inset) is divergent but integrable at the origin.

For $n = 4$, we present 2 pairs of curves in Figure 7:

- $P_4^{(-)}(r, C = 9)$ (continuous line) and $P_4^{(-)}(r, C = 3)$ (thin broken line) which go smoothly to zero at the origin;
- $P_4^{(-)}(r, C = 6)$ (bold broken line) and $P_4^{(-)}(r, C = 2)$ (dotted line) which diverge at $r = 0$ but remain integrable in this neighborhood. This curves are magnified in insets in the lower part of Figure 7.

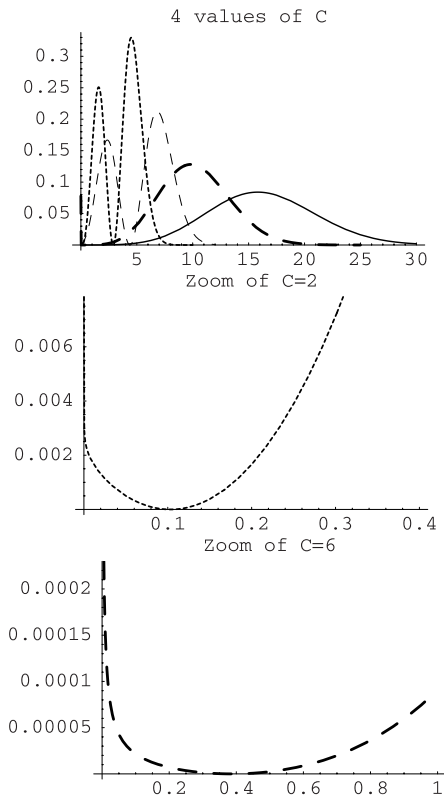


Fig. 7. Probability distributions $P_4^{(-)}(r, C)$ for $n = 4$.

The $P_{3,4}^{(-)}(r, C)$ are computed with the expression of the wave function in equation (8). Since the form of the special Biconfluent Heun Polynomials $H_{3,4}(\xi, C)$ are quite involved, we shall not give here the explicit form of $P_{3,4}^{(-)}(r, C)$.

Higher stationary states come also in pairs $n = 5, 6$ and $n = 7, 8$ with algebraic relations of third and fourth order in C^2 (and/or l). The angular momentum l will have respectively three and four values and a more complicated structure in terms of magnetic field arises. General features concerning two classes of smooth and singular behavior at the origin however remain. We shall discuss them in a more extensive study later on. Beyond these values of n , one can only tackle the problem numerically since the algebraic equations of higher order do not possess solutions in close form.

4 Comments and conclusion

The presence of fractional power singularities of the wave functions at the origin of coordinates calls for some comments. The rise of the probability of finding a separation r between two identical particles within dr suggests a tendency for the particles to cluster together. Since $P_n^{(-)}(r, C)dr \simeq r^{-\lambda}dr \simeq d(r^{1-\lambda})$ and as $0 < \lambda < 1$, integration of r from 0 to any value $r = a$ close to the origin is finite. This means that the probability of finding any r in the interval $[0, a]$ is always finite, despite the

Table 1. Table of closest zeros r_0 of low-lying probability distributions with given C , energy levels and the locations of the classically limiting points r_1 at these values of the energy ($\epsilon_{n,l_k}^{(-)} = n + 1 - 2|l_k|$). Values of the magnetic B in tesla, corresponding to chosen values of C are listed in the second column.

n	B	C	$\frac{r_0}{l_0}$	$\epsilon_{n,l_k}^{(-)}$	$\frac{r_1}{l_0}$
1	940215	1	0	3	0.3409025
2	470108	$\sqrt{2}$	0.207107	1.75	0.5
	117527	$2\sqrt{2}$	0	0.625	0.380453
3	940215	1	0	6.60434	0.215634
	37608.6	5	0	0.263969	3.84573
4	117527	$2\sqrt{2}$	0.322087	0.545623	1.99029
	235054	2	0.1032474	1.9443	0.666414
4	104468	3	0	0.965468	1.07414
	26117.1	6	0.386699	0.184408	5.4233
	11607.6	9	0	0.120639	9.1545

apparent divergence. Moreover the clustering tendency in the presence of classical Coulomb repulsion is somewhat counter-intuitive. To reconcile this behavior with quantum mechanics we have given in Table 1 the values of the zeros r_0 of the probability distributions nearest to zero, from which start divergent behaviors for various n and l_k as well as the locations r_1 below which classical motion is not possible in corresponding effective potentials. For the four low lying n , we see that always $r_0 < r_1$. Hence the divergence occurs only in forbidden classical regions of the motion, thus not directly observable. Anyhow, such situations occur also at tremendously high B -field, of the order of 10^5 tesla.

Finally, it is amusing to note that there is an analog of this situation in electrostatics. The electric field, near the edge of a two-dimensional wedge of opening angle θ is the analog of $\psi_{n,l_k}^{(\sigma)}(\theta, r)$ and behaves as $r^{\frac{\pi}{\theta}-1}$, r being the distance from the edge [6]. Thus it diverges for $\pi < \theta < 2\pi$. Now the electric field energy density is the analog of $P_n^{(-)}(r, C)$, but here it has always a nice behavior $r^{2\frac{\pi}{\theta}-1}$ for $0 < \theta < 2\pi$.

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